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Cubic hyper-resolutions of analytic varieties with hypersurface ordinary singularities of dimension ≤ 5

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§1 Hypersurface ordinary singularities of dimension ≤ 5

Let $(Y^n, o) \subset (\mathbb{C}^{n+1}, o)$ be a pure-dimensinal hypersurface germ and let $\nu : (X^n, \nu^{-1}(o)) \to (Y^n, o)$ be the normalization map. We define $f := \iota \circ \nu : (X^n, f^{-1}(o)) \to (\mathbb{C}^{n+1}, o)$, where $\iota : (Y^n, o) \subset (\mathbb{C}^{n+1}, o)$ is the inclusion map.

1.1 Definition. we say (Y^n, o) is an *ordinary singularity* if
(i) $(X^n, f^{-1}(o))$ is non-singular, and
(ii) $f := \iota \circ \nu : (X^n, f^{-1}(o)) \to (\mathbb{C}^{n+1}, o)$ is *simultaneously stable*, i.e., small deformation of the multi-germ f of a holomorphic map is *trivial*.

For an ordinary singularity $(Y^n, o) \subset (\mathbb{C}^{n+1}, o)$ with $f := \iota \circ \nu : (X^n, f^{-1}(o)) \to (\mathbb{C}^{n+1}, o)$ being the same as above, we put

$f^{-1}(o) := \{p_1, p_2 \cdots, p_k\}$,

$R(f)_{p_i} := \mathcal{O}_{X,p_i} / f^* \mathfrak{m}_o \cdot \mathcal{O}_{X,p_i}$

$(1 \leq i \leq k,\ \mathfrak{m}_o$ is the maximal ideal of $\mathcal{O}_{\mathbb{C}^{n+1},o})$, and

$C_i := \{q \in X^n | R(f)_q \simeq R(f)_{p_i}\}$ (*contact class* of f at p_i).

1.2 Proposition([8, Proposition 7.1], [5]). *$f := \iota \circ \nu : (X^n, f^{-1}(o)) \to (\mathbb{C}^{n+1}, o))$ is simultaneously stable iff both of the following conditions are satisfied:*
(i)$f_i := f_{|p_i} : (X^n, p_i) \to (\mathbb{C}^{n+1}, o)$ is stable for any i $(1 \leq i \leq k)$,
(ii) $(df)_{p_1}(T_{C_1,p_1}), \cdots, (df)_{p_k}(T_{C_k,p_k})$ are in genaral position in $T_{\mathbb{C}^{n+1},o}$, where T_{C_i,p_i} denotes the tangent space of C_i at p_i and so on.

1.3 Proposition([7]). *Let $f : (\mathbb{C}^n, o) \to (\mathbb{C}^m, o)$ be a holomorphic map germ. Assume that (i) $(n, m) \in \{$nice range$\}$(cf. [6]), (ii) $n < m$, and (iii) $n \leq \frac{2}{3}m + 1$. Then f is stable iff $R(f)_o$ is isomorphic to one of the following $\mathbb{C}$-algebras:*

$$A_0 := \mathbb{C}[[x]]/(x), \quad A_1 := \mathbb{C}[[x]]/(x^2), \quad A_2 := \mathbb{C}[[x]]/(x^3).$$

When $n < m$, the normal forms of holomorphic maps f with $R(f)_o \simeq A_\ell$ $(0 \leq \ell \leq 2)$ are given as follows (cf. [4]):

(i)In the case of $R(f)_o \simeq A_0$:

$$\begin{cases} y_i \circ f = x_i \ (1 \leq i \leq n) \\ y_i \circ f = 0 \ (n+1 \leq i \leq m), \end{cases}$$

(ii)In the case of $R(f)_o \simeq A_1$:

$$\begin{cases} y_i \circ f = x_i \ (1 \leq i \leq n-1) \\ y_n \circ f = x_n^2 \\ y_{n+i} \circ f = x_i x_n \ (1 \leq i \leq m-n \leq n-1), \end{cases}$$

(iii)In the case of $R(f)_o \simeq A_2$:

$$\begin{cases} y_i \circ f = x_i \ (1 \leq i \leq n-1) \\ y_n \circ f = x_n^3 + x_1 x_n \\ y_{n+i} \circ f = x_{2i} x_n + x_{2i+1} x_n^2 \ \ (1 \leq i \leq m-n, \ \ 2(m-n)+1 \leq n-1). \end{cases}$$

1.4 Remark. When $m = n+1$, (n, m) satisfies the conditions (i), (ii) and (iii) in Theorem 1.3 iff $1 \leq n \leq 5$, and the case (iii) above occurs only when $n = 4, 5$.

Using these facts, we can calculate the defining equations of ordinary singularities of dimension≤ 5.

1.5 Proposition([8]). *The defining equations of hypersurface ordinary singularities of dimension $n \leq 5$ in $\mathbb{C}^{n+1}$ are given as follows:*

$i)n = 1:$
$a)_1 \ y_1 = 0$
$a)_2 \ y_1 y_2 = 0$

$ii)n = 2:$
$a)_k \ \ y_1 \cdots y_k = 0 \ (1 \leq k \leq 3)$
$b) \ y_1^2 - y_2^2 y_3 = 0$

$iii)n = 3:$
$a)_k \ \ y_1 \cdots y_k = 0 \ (1 \leq k \leq 4)$
$b) \ y_1^2 - y_2^2 y_3 = 0$
$a)_1 + b) \ y_4(y_1^2 - y_2^2 y_3) = 0$

$iii)n = 4:$
$a)_k \ \ y_1 \cdots y_k = 0 \ (1 \leq k \leq 5)$
$b) \ \ y_1^2 - y_2^2 y_3 = 0$
$a)_1 + b) \ \ y_4(y_1^2 - y_2^2 y_3) = 0$
$a)_2 + b) \ \ y_4 y_5(y_1^2 - y_2^2 y_3) = 0$
$c) \ \ y_5^3 + 2y_1 y_3 y_5^2 + (y_1^2 y_3^2 - 3y_2 y_3 y_4 + y_1 y_2^2) y_5 - \{y_3^3 y_4 + y_2(y_2^2 + y_1 y_3^2)\} y_4 = 0$

$iv)n = 5:$
$a)_k \ \ y_1 \cdots y_k = 0 \ (1 \leq k \leq 6)$
$b) \ \ y_1^2 - y_2^2 y_3 = 0$
$a)_1 + b) \ \ y_4(y_1^2 - y_2^2 y_3) = 0$
$a)_2 + b) \ \ y_4 y_5(y_1^2 - y_2^2 y_3) = 0$
$a)_3 + b) \ \ y_4 y_5 y_6(y_1^2 - y_2^2 y_3) = 0$
$b) + b) \ \ (y_1^2 - y_2^2 y_3)(y_4^2 - y_5^2 y_6) = 0$

$c)\ y_5^3 + 2y_1y_3y_5^2 + (y_1^2y_3^2 - 3y_2y_3y_4 + y_1y_2^2)y_5 - \{y_3^3y_4 + y_2(y_2^2 + y_1y_3^2)\}y_4 = 0$
$a)_1 + c)\ y_6[y_5^3 + 2y_1y_3y_5^2 + (y_1^2y_3^2 - 3y_2y_3y_4 + y_1y_2^2)y_5 - \{y_3^3y_4 + y_2(y_2^2 + y_1y_3^2)\}y_4] = 0$

PROOF: For example, we shall show how the equation iii) c) follows. In the case of n=4, the normal form of a holomorphic map f with $R(f)_o \simeq A_2$ is given by

$$(1.1)\qquad \begin{cases} y_i \circ f = x_i \ (1 \le i \le 3) \\ y_4 \circ f = x_4^3 + x_1x_4 \\ y_5 \circ f = x_2x_4 + x_3x_4^2. \end{cases}$$

Substituting $x_1 = y_1, x_2 = y_2, x_3 = y_3$ into the last two equations, we have

$$\begin{cases} x_4^3 + y_1x_4 - y_4 = 0 \\ y_3x_4^2 + y_2x_4 - y_5 = 0. \end{cases}$$

We regard this as a simultaneous equation for x_4 with coefficients in the polynomial ring $\mathbb{C}[y_1, \cdots, y_5]$. To eliminate x_4, we calculate the *resultant*

$$\begin{vmatrix} 1 & 0 & y_1 & -y_4 & 0 \\ 0 & 1 & 0 & y_1 & -y_4 \\ y_3 & y_2 & -y_5 & 0 & 0 \\ 0 & y_3 & y_2 & -y_5 & 0 \\ 0 & 0 & y_3 & y_2 & -y_5 \end{vmatrix}$$

of the equation(cf. [12, Chapter 11]). Then we get the equation iii) c) in the proposition. For more details, see [8].

Q.E.D.

§2 Cubic hyper-resolutions of hypersurface ordinary singularities of dimension ≤ 5

We denote by $\mathbb{Z}$ the integer ring.

2.1 Definition. For $n \in \mathbb{Z}$ with $n \ge 0$ the *augmented n-cubic category*, denoted by $\square_n^+$, is defined to be a category whose objects $\mathrm{Ob}(\square_n^+)$ and the set of homomorphisms $\mathrm{Hom}_{\square_n^+}(\alpha, \beta)$ $(\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n), \beta = (\beta_0, \beta_1, \cdots, \beta_n) \in \mathrm{Ob}(\square_n^+))$ are given as follows:

$$\mathrm{Ob}(\square_n^+) := \{\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^{n+1} \mid 0 \le \alpha_i \le 1 \text{ for } 0 \le i \le n\},$$

$$\mathrm{Hom}_{\square_n^+}(\alpha, \beta) := \begin{cases} \alpha \to \beta \text{ (an arrow from } \alpha \text{ to } \beta) & \text{if } \alpha_i \le \beta_i \text{ for } 0 \le i \le n \\ \emptyset & \text{otherwise.} \end{cases}$$

For $n = -1$ we understand $\square^+_{-1}$ to be the punctual category $\{*\}$, i.e., the category consisting of a single point.

Notice that $\mathrm{Ob}(\square^+_n)$ can be considered as a finite ordered set whose order is defined by $\alpha \leq \beta \Longleftrightarrow \alpha \to \beta$ for $\alpha, \beta \in \mathrm{Ob}(\square^+_n)$.

2.2 Definition. A $\square^+_n$-*analytic variety* is defined to be a contravariant functor $X.$ from $\square^+_n$ to the category of complex analytic varieties $(\mathrm{An}/\mathbb{C})$. It is also called an *augmented n-cubic analytic variety*.

2.3 Definition. Let $X., Y.$ be $\square^+_n$-analytic varieties. We define a morphism $\Phi. : X. \longrightarrow Y.$ to be a natural transformation from the functor $X.$ to the one $Y.$ over the identity functor $\mathrm{id} : \square^+_n \longrightarrow \square^+_n$.

2.4 Definition. For a $\square^+_n$-analytic variety $X.$, a contravariant functor $Y.$ from $\square^+_1$ to the category of $\square^+_n$-analytic varieties is called a *2-resolution of* $X.$ if $Y.$ is defined by a cartesian square of morphisms of $\square^+_n$-analytic varieties

$$\begin{array}{ccc} Y_{11.} & \longrightarrow & Y_{01.} \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y_{10.} & \longrightarrow & Y_{00.}, \end{array} \tag{2.1}$$

which satisfies the following conditions:

(i) $Y_{00.} = X.$,

(ii) $Y_{01.}$ is a smooth $\square^+_n$-analytic variety, i.e., a contravariant functor from $\square^+_n$ to the category of smooth analytic varieties,

(iii) the horizontal arrows are closed immersion of $\square^+_n$-analytic varieties

(iv) f is a proper morphism between $\square^+_n$-analyitc varieties, and

(v) f induces an isomorphism from $Y_{01\beta} - Y_{11\beta}$ to $Y_{00\beta} - Y_{10\beta}$ for any $\beta \in Ob(\square^+_n)$.

We think of the cartesian square in (2.1) as a morphism from the $\square^+_{n+1}$-complex analytic variety $Y_{1..}$ to the one $Y_{0..}$ and write it as $Y_{1..} \longrightarrow Y_{0...}$ For a 2-resolution $Z.$ of $Y_{1..}$, we define the $\square^+_{n+3}$-analytic variety $rd(Y., Z.)$ by

$$rd(Y., Z.) := \begin{array}{ccc} Z_{11\cdot} & \longrightarrow & Z_{01\cdot} \\ \downarrow & & \downarrow \\ Z_{10\cdot} & \longrightarrow & Y_{0...} \end{array}$$

and call it the *reduction* of $\{Y., Z.\}$.

2.5 Definition. Let X be an analytic variety and let $\{X.^1, X.^2, \cdots, X.^n\}$ be a sequence of $\square^+_r$-analytic varieties $X.^r$ $(1 \leq r \leq n)$ such that

(i) $X_{\cdot}^{1}$ is a 2-resolution of X,

(ii) $X_{\cdot}^{r+1}$ is a 2-resolution of $X_{1\cdot}^{r}$ for $1 \leq r \leq n-1$.

Then, by induction on n, we define

$$Z. := rd(X_{\cdot}^{1}, X_{\cdot}^{2}, \cdots, X_{\cdot}^{n}) := rd(rd(X_{\cdot}^{1}, X_{\cdot}^{2}, \cdots, X_{\cdot}^{n-1}), X_{\cdot}^{n}).$$

With this notation, if Z_{α} are smooth for all $\alpha \in \square_n$, we call $Z.$ an *augmented n-cubic hyper-resolution* of X.

2.6 Definition. We call the cartesian square in (2.1) the *2-resolution of X. by normalization* if it satisfies that (i) $f : Y_{01\cdot} \longrightarrow Y_{00\cdot}$ is the normalization, (ii) $Y_{10\cdot}$ is the discriminant of f (i.e., the smallest, closed $\square_n^+$- analytic variety of $Y_{00\cdot}$ with $Y_{01\cdot} - f^{-1}(Y_{10\cdot}) \simeq Y_{00\cdot} - Y_{10\cdot}$), and (iii)$Y_{11\cdot} = f^{-1}(Y_{10\cdot})$.

2.7 Definition. Let X be an analytic variety. If there exists an augmented n-cubic hyper-resolution $Z. := rd(X_{\cdot}^{1}, X_{\cdot}^{2}, \cdots, X_{\cdot}^{n})$ of X such that $X_{\cdot}^{r+1}$ is the 2-resolution of $X_{1\cdot}^{r}$ by normalization for every r with $0 \leq r \leq n-1$ (we understand $X_{1\cdot}^{0} = X$), then we say that an augmented cubic hyper-resolution of X is obtained by *successive normalizations*.

2.8 Example. Let $(Y, o) \subset (\mathbb{C}^5, o)$ be the hypersurface ordinary singularity defined by the equation iii) c) in Proposition 1.5. We shall show that an augmented cubic hyper-resolution of Y is obtained by *successive normalizations*. The map $f := \iota \circ \nu : (\mathbb{C}^4, o) \longrightarrow (\mathbb{C}^5, o)$, the composite of the normalization $\nu : (\mathbb{C}^4, o) \longrightarrow (Y, o)$ and the inclusion $\iota : (Y, o) \subset (\mathbb{C}^5, o)$, is given by (1.1). Let

$$D_i(f) := \{x \in \mathbb{C}^4 | \ \sharp f^{-1}(f(x)) \geq i\}, \quad (i = 2, 3)$$

denote the i-ple point locus of f and let

$$D_i(Y) := \{y \in Y | \ \mu_y(Y) \geq i\}, \quad (i = 2, 3)$$

denote that of Y, where $\mu_y(Y)$ is the *multiplicity* of Y at $y \in Y$. By calculation, we can see that each of these loci is defined by the following equation:

$$D_2(f) : \ x_2^2 + (x_3 x_4)x_2 + x_3^2(x_4^2 + x_1) = 0 \tag{2.2}$$

(this equation follows from that a point $p \in \mathbb{C}^4$ belongs to $D_2(f)$ iff the equation $f(p) = f(x)$ has other roots than p),

$$D_3(f) : \ x_2 = x_3 = 0,$$

$$D_2(Y) : \ y_3 y_5 + y_2^2 + y_1 y_3^2 = y_2 y_5 + y_3^2 y_4 = 0 \tag{2.3}$$

(since $D_2(Y) = f(D_2(f))$, we obtain this by eliminating $x_1, \cdots, x_4$ from the equation of $D_2(f)$ in (2.2) and the equation in (1.1)),

$$D_3(Y) : \ y_2 = y_3 = y_5 = 0.$$

Note that $Sing(D_2(f)) = D_3(f)$ and $Sing(D_2(Y)) = D_3(Y)$, where $Sing(Z)$ denotes the singular locus of $Z = D_2(f), D_2(Y)$. We define a $\square_1^+$-analytic variety $X_{\cdot}^{1}$ to be

$$
\begin{tikzcd}
X^1_{11} := D_2(f) \arrow[r, "j_1"] \arrow[d, "\mu_1 := \nu_1|D_2(f)"'] & \mathbb{C}^4 =: X^1_{01} \arrow[d, "\nu_1"] \\
X^1_{10} := D_2(Y) \arrow[r, "i_1"'] & Y =: X^1_{00},
\end{tikzcd}
\tag{2.4}
$$

where $\nu_1 := \nu$, the normalization of Y, and the horizontal arrows are inclusions. This diagram is nothing but the 2-resolution of Y by *normalization*. We regard the map $\mu_1 : D_2(f) \longrightarrow D_2(Y)$ as a $\square_0^+$-analytic variety. We are now going to show that a 2-resolution of the $\square_0^+$-analytic variety $\mu_1 : D_2(f) \longrightarrow D_2(Y)$ is also obtained by *normalization*.

Step(1) First, we shall show that the strict transform $D_2(Y)^*$ of $D_2(Y)$ by the blowing-up $\sigma : \hat{\mathbb{C}}^5 \longrightarrow \mathbb{C}^5$ of $\mathbb{C}^5$ with non-singualr center $D_3(Y)$ becomes non-singular, and that the restriction map $\nu_{20} := \sigma_{|D_2(Y)^*} : D_2(Y)^* \longrightarrow D_2(Y)$ is the normalization. We put

$$
\begin{aligned}
g_1 &:= y_3y_5 + y_2^2 + y_1y_3^2, \\
g_2 &:= y_2y_5 + y_3^2y_4
\end{aligned}
$$

(cf. (2.3)) and let $\mathcal{I}_{D_2(Y)}$ denote the ideal sheaf of $D_2(Y)$ in $\mathcal{O}_{\mathbb{C}^5}$, which is generated by g_1 and g_2 as a $\mathcal{O}_{\mathbb{C}^5}$-module. Here we should note that Buchberger's algorithm to compute the *(reduced) Groebner basis* of an ideal of the polynomial ring (cf. [2, Chapter 2, §7]) works as well for computing the *standard basis* (cf. [1, Corollary 4.2.1]) of an ideal of the convergent power series ring. Hence, applying this algorithm to $\{g_1, g_2\}_o := \mathcal{I}_{D_2(Y),o}$, the stalk of $\mathcal{I}_{D_2(Y)}$ at the origin $o \in \mathbb{C}^5$, we can find that g_1, g_2 and

$$
g_3 := y_2^3 - y_3^3y_4 + y_1y_2y_3^2
$$

constitute the standard basis of $\mathcal{I}_{D_2(Y),o}$. Since $\mu_{D_3(Y),o}(g_i) = \mu_o(g_i), i = 1,2,3,$ where $\mu_{D_3(Y),o}(g_i)$ denotes the multiplicity of g_i along $D_3(Y)$ at the origin $o \in \mathbb{C}^5$, which is defined to be the largest μ such that the germ of g_i at o belongs to $(\mathcal{I}_{D_3(Y),o})^\mu$, the stalk $\mathcal{I}_{D_2(Y)^*,x}$ of the ideal sheaf $\mathcal{I}_{D_2(Y)^*}$ at $x \in \sigma^{-1}(o)$ is generated by the strict transforms g_i^* of g_i, i=1, 2, 3, by σ as a $\mathcal{O}_{\hat{\mathbb{C}}^5,o}$-module ([1, Lemma 7.1]). In fact, calculating in terms of local coordinates, we can see that $\mathcal{I}_{D_2(Y)^*,x}$, $x \in \sigma^{-1}(o)$, is generated by g_1^*, g_2^* since $g_3 = y_2g_1 - y_3g_2$. The blowing-up $\sigma : \hat{\mathbb{C}}^5 \longrightarrow \mathbb{C}^5$ of $\mathbb{C}^5$ with non-singular center $D_3(Y) : y_2 = y_3 = y_5 = 0$ is explicitly described as follows:

$$
\hat{\mathbb{C}}^5 := \{(y_1, \cdots, y_5) \times (\xi_2 : \xi_3 : \xi_5) \in \mathbb{C}^5 \times \mathbb{P}^2 \mid y_i\xi_j - y_j\xi_i = 0, i, j = 2, 3, 5\},
$$

$\sigma := \mathrm{Pr}_{\mathbb{C}^5|\hat{\mathbb{C}}^5} : \hat{\mathbb{C}}^5 \longrightarrow \mathbb{C}^5$, the restriction of the projection $\mathrm{Pr}_{\mathbb{C}^5} : \mathbb{C}^5 \times \mathbb{P}^2 \longrightarrow \mathbb{C}^5$ to $\hat{\mathbb{C}}^5$.

Let $U_i, i = 2, 3, 5$, denote the open subset of $\hat{\mathbb{C}}^5$ defined by $\xi_i \neq 0$. On U_3, we can take $(y_1, y_3, y_4, u_2, u_5)$ as a local coordinate system, where $u_i = \frac{\xi_i}{\xi_3}, i = 2, 5$, and σ is written as

$$\sigma : (y_1, y_3, y_4, u_2, u_5) \longrightarrow (y_1, y_3 u_2, y_3, y_4, y_3 u_5) = (y_1, y_2, y_3, y_4, y_5)$$

in term of this local coordinate system. Hence the strict transforms g_1^*, g_2^* of g_1, g_2 by σ is given by

$$\begin{aligned} g_1^* &= y_3^{-2}(\sigma^{-1}(g_1)) = u_5 + u_2^2 + y_1, \\ g_2^* &= y_3^{-2}(\sigma^{-1}(g_2)) = y_4 + u_2 u_5. \end{aligned} \tag{2.5}$$

Since the rank of the Jacobian matrix $\partial(g_1^*, g_2^*)/\partial(y_1, y_3, y_4, u_2, u_5)$ is maximal throughout $D_2(Y)^* \cap U_3$, we conclude that $D_2(Y)^*$ is non-singular in U_3. By (2.5), the map $\nu_{20|D_2(Y)^* \cap U_3} : D_2(Y)^* \cap U_3 \longrightarrow D_2(Y) \cap \sigma(U_3)$ is obviously a finite map. Therefore, $\nu_{20|D_2(Y)^* \cap U_3} : D_2(Y)^* \cap U_3 \longrightarrow D_2(Y) \cap \sigma(U_3)$ is nothing but the normalization, since ν_{20} gives rise to an isomorphism between $D_2(Y)^* \cap U_3 - \sigma^{-1}(D_3(Y))$ and $D_2(Y) \cap \sigma(U_3) - D_3(Y)$. On other $U_i, i = 2, 5$, we can also check that $D_2(Y)^* \cap U_i$ is non-singular and the map $\nu_{20|D_2(Y)^* \cap U_i}$: $D_2(Y)^* \cap U_i \longrightarrow D_2(Y) \cap \sigma(U_i)$ is the normalization. Hence the map ν_{20} : $D_2(Y)^* \longrightarrow D_2(Y)$ is the non-singular normalization of $D_2(Y)$.

Step(2) Secondly, we shall show that the normalization of $D_2(f)$ is non-singular. The defining equation of $D_2(f)$ in (2.2) is transformed as follows:

$$\begin{aligned} & x_2^2 + (x_3 x_4) x_2 + x_3^2 (x_4^2 + x_1) \\ &= \{x_2 + \tfrac{1}{2} x_3 x_4 + \sqrt{-x_1 - \tfrac{3}{4} x_4^2} \cdot x_3\}\{x_2 + \tfrac{1}{2} x_3 x_4 - \sqrt{-x_1 - \tfrac{3}{4} x_4^2} \cdot x_3\} \\ &= (z + \sqrt{x} y)(z - \sqrt{x} y) = z^2 - x y^2, \end{aligned}$$

where $x := -x_1 - \frac{3}{4} x_4^2$, $y := x_3$, $z := x_2 + \frac{1}{2} x_3 x_4$. Note that $D_3(f)$ is given by $y = z = 0$. The map $\nu_{21} : D_2(f)^* := \mathbb{C}^3 \longrightarrow D_2(f) \subset \mathbb{C}^4$ defined by $(u, v, x_4) \longrightarrow (u^2, v, uv, x_4) = (x, y, z, x_4)$ is the normalization of $D_2(f)$, since ν_{21} gives rise to an isomorphism between $D_2(f)^* - \{v = o\}$ and $D_2(f) - D_3(f)$. Therefore the normalization of $D_2(f)$ is non-singular

Step(3) We consider the following diagram:

$$\begin{array}{ccc} D_2(Y)^* & \xleftarrow{\tilde{\mu_1}} & D_2(f)^* \\ {\scriptstyle \nu_{20}} \downarrow & & \downarrow {\scriptstyle \nu_{21}} \\ D_2(Y) & \xleftarrow[\mu_1]{} & D_2(f), \end{array}$$

where $\tilde{\mu_1}$ is the lifting of μ_1. This gives the normalization of $\Box_0^+$-analytic variety $\mu_1 : D_2(f) \longrightarrow D_2(Y)$ and, further, gives rise to the following 2-resolution of it:

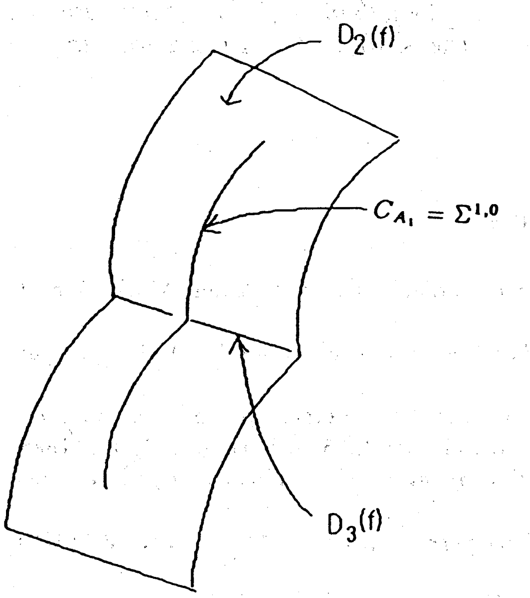
$$
\begin{tikzcd}
 & X^2_{111} := \nu_{21}^{-1}(D_3(f)) \arrow[rr, "j_{21}"] \arrow[dl, "\tilde{\lambda}_1"'] \arrow[dd, "\mu_{21}"] & & D_2(f)^* =: X^2_{011} \arrow[dl, "\widetilde{\mu_1}"'] \arrow[dd, "\nu_{21}"] \\
X^2_{110} := \nu_{20}^{-1}(D_3(Y)) \arrow[rr, "j_{20}"] \arrow[dd, "\mu_{20}"'] & & D_2(Y)^* =: X^2_{010} \arrow[dd, "\nu_{20}"'] & \\
 & X^2_{101} \quad D_3(f) \arrow[rr, "i_{21}"] \arrow[dl, "\lambda_1"] & & D_2(f) =: X^2_{001} = X^1_{11} \arrow[dl, "\mu_1"] \\
X^2_{100} := D_3(Y) \arrow[rr, "i_{20}"'] & & D_2(Y) =: X^2_{000} = X^1_{10}, &
\end{tikzcd}
\tag{2.6}
$$

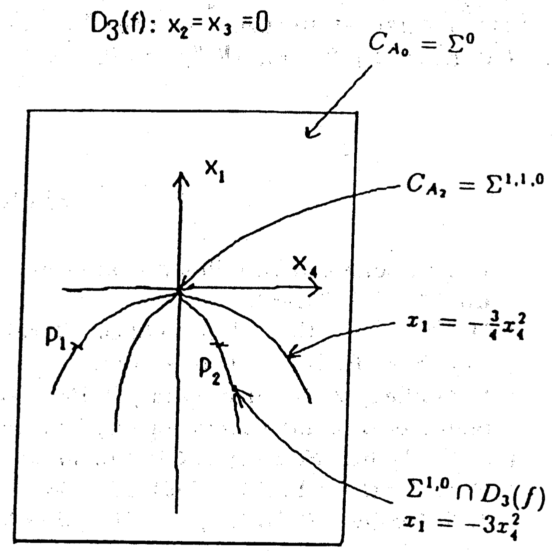
where $i_{2\alpha}, j_{2\alpha}, \alpha = 0, 1$, are inclusion maps, and $\lambda_1 := \mu_{1|D_3(f)}$, $\tilde{\lambda}_1 := \tilde{\mu}_{1|\nu_{21}^{-1}(D_3(f))}$. Since all of $D_2(Y)^*$, $D_2(f)^*$, $D_3(Y)$, $D_3(f)$, $\nu_{20}^{-1}(D_3(Y))$ and $\nu_{21}^{-1}(D_3(f))$ are non-singular, replacing $\mu_1 : D_2(f) \to D_2(Y)$ by $\nu_1 : \mathbb{C}^4 \to Y$ (this means to form the *reduction* of (2.4) and (2.6)), we obtain the following cubic hyper-resolution of Y:

$$
\begin{tikzcd}
 & \nu_{21}^{-1}(D_3(f)) \arrow[rr, "j_{21}"] \arrow[dl, "\tilde{\lambda}_1"'] \arrow[dd, "\mu_{21}"] & & D_2(f)^* \arrow[dl, "\tilde{\mu}_1"'] \arrow[dd, "j_1 \circ \nu_{21}"] \\
\nu_{20}^{-1}(D_3(Y)) \arrow[rr, "j_{20}"] \arrow[dd, "\mu_{20}"'] & & D_2(Y)^* \arrow[dd, "i_1 \circ \nu_{20}"'] & \\
 & D_3(f) \arrow[rr, "j_1 \circ i_{21}"'] \arrow[dl, "\lambda_1"] & & \mathbb{C}^4 \arrow[dl, "\nu_1"] \\
D_3(Y) \arrow[rr, "i_1 \circ i_{20}"'] & & Y. &
\end{tikzcd}
$$

Therefore we conclude that an augmened hyper-resolution of Y is obtained by *successive normalizations*.

2.9 Remark. The various singular point loci of the map f in Example 2.8 are described as follows:





(*) p_1, p_2 are together pushed down to the singularity $w(xy^2 - z^2) = 0$ in $\mathbf{C}^5$.

$$C_{A_i} := \{\ x \in \mathbf{C}^4 \mid R(f)_x \simeq A_i\}\ \ (i = 0, 1, 2),$$
$$\Sigma^i := \{\ x \in \mathbf{C}^4 \mid \dim Ker\, df_x = i\ \}\ \ (i = 0, 1),$$
$$\Sigma^{1,i} := \{\ x \in \Sigma^1 \mid \dim Ker\, d(f_{|\Sigma^1})_x = i\ \}\ \ (i = 0, 1),$$
$$\Sigma^{1,1,0} := \{\ x \in \Sigma^{1,1} \mid \dim Ker\, d(f_{|\Sigma^{1,1}})_x = 0\ \}.$$

2.10 Theorem. *Let X be an analytic variety with hypersurface ordinary singularities of dimension ≤ 5, then a cubic hyper-resolution of X is obtained by successive normalizations.*

PROOF: In the similar manner to prove Proposition 2.15 in [10, I], we can prove this, using the calculation in Example 2.8 above.

Q.E.D.

As a by-product, we obtain the following.

2.11 Corollary([10],[11]). *Let $\pi : \mathfrak{X} \longrightarrow M$ be a locally trivial family of compact complex projective varieties with hypersurface ordinary singularities of dimension ≤ 5, parametrized by a complex manifold M. We define $R^\ell_{\mathbf{Z}}(\pi) := R^\ell\pi_*\mathbf{Z}_{\mathfrak{X}}$ (modulo torsion) $(0 \leq \ell \leq 2(\dim\mathfrak{X}\text{-}\dim M))$, $R^\ell_{\mathbf{Q}}(\pi) := R^\ell_{\mathbf{Z}}(\pi) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $R^\ell_{\mathcal{O}}(\pi) := R^\ell\pi_*(\pi^\cdot\mathcal{O}_M) \simeq \mathbb{R}^\ell\pi_*(DR^\cdot_{\mathfrak{X}/M})$, where $\pi^\cdot\mathcal{O}_M$ is the topological inverse of the structure sheaf of M by the map $\pi : \mathfrak{X} \longrightarrow M$ and $DR^\cdot_{\mathfrak{X}/M}$ the* *cohomological relative de Rham complex of the family* $\pi : \mathfrak{X} \to M$. *Then there exist a family of increasing sub-local systems* $\mathbb{W}$ *(weight filteration) on* $R^\ell_{\mathbb{Q}}(\pi)$ *and a family of decreasing holomorphic subbundles* $\mathbb{F}$ *(Hodge filteration) on* $R^\ell_{\mathcal{O}}(\pi)$ *such that* $\{R^\ell_{\mathbb{Z}}(\pi), (R^\ell_{\mathbb{Q}}(\pi), \mathbb{W}[\ell]), (R^\ell_{\mathcal{O}}(\pi), \mathbb{W}[\ell], \mathbb{F})\}$ *is a variation of mixed Hodge structure , where* $\mathbb{W}[\ell]$ *denotes the shift of the filteration degree to the right by* ℓ, *i.e.,* $\mathbb{W}[\ell]_q := \mathbb{W}_{q-\ell}$.

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